EXERCISE – IV HINTS & SOLUTIONS

Sol.1 $\int \frac{\tan 2\theta}{\sqrt{(\cos^2\theta)^3 + (\sin^2\theta)^3}}.d\theta$

$I = \int \frac{\sin 2\theta}{\cos 2\theta \sqrt{3+\cos^2 2\theta}}.d\theta$

Let $\cos 2\theta = t$

$I = -\int \frac{dt}{t\sqrt{3+t^2}}$

Now let $\frac{1}{t} = w$

Sol.2 $I = \int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$

$= \int \frac{x^4(5+4x)}{x^{10}\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}dx$

$= \int \frac{\frac{5}{x^6}+\frac{4}{x^5}}{\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}dx$

put $1+\frac{1}{x^4}+\frac{1}{x^5} = t \Rightarrow \left(\frac{4}{x^5}+\frac{5}{x^6}\right)dx = -dt$

$= -\int \frac{dt}{t^2} = \frac{1}{t} + c = \frac{1}{1+\frac{1}{x^4}+\frac{1}{x^5}} + c$

$I = \frac{x^5}{x^5+x+1} + c$

Sol.3 $I = \int \frac{\cos^2 x}{1+\tan x}dx$

$= \int \frac{\cos^2 x.\cos x}{(\cos x+\sin x)}dx$

$= \int \frac{\cos^2 x.\cos x(\cos x-\sin x)}{(\cos^2 x-\sin^2 x)}dx$

$I = \int \frac{\cos^2 x(\cos^2 x-\sin x\cos x)}{(\cos^2 x-\sin^2 x)}dx$

$= \frac{1}{4}\int\left(\frac{1+\cos 2x}{\cos 2x}\right)(2\cos^2 x - 2\sin x\cos x)\,dx$

$= \frac{1}{4}\int\left(\frac{1+\cos 2x}{\cos 2x}\right)(1+\cos 2x-\sin 2x)\,dx$

$= \frac{1}{4}\int\left[\frac{(1+\cos 2x)^2}{\cos 2x} - \left(\frac{1+\cos 2x}{\cos 2x}\right)\sin 2x\right]dx$

put $2x = t \Rightarrow dx = \frac{dt}{2}$

$= \left[\frac{1}{8}\int\frac{(1+\cos t)^2}{\cos t}dt - \frac{1}{8}\left(\frac{1+\cos t}{\cos t}\right)\sin t\,dt\right]$

$\cos t = z \Rightarrow \sin t\,dt = -dz$

$= \frac{1}{8}\int\left(\frac{1+\cos^2 t+2\cos t}{\cos t}\right)dt + \frac{1}{8}\int\left(\frac{1+z}{z}\right)dz$

$= \frac{1}{8}\int(\sec t+\cos t+2)\,dt + \frac{1}{8}\ell n\, z + z + c$

$= \frac{1}{8}$ [ℓn |sec t + tan t| + sin t + 2t] $+ \frac{1}{8}\ell n(\cos t) + \cos t + c$

$= \frac{1}{4}\ \ell n(\cos x+\sin x) + \frac{x}{2} + \frac{1}{8}(\sin 2x+\cos 2x) + c$

Sol.4 $I = \int \frac{dx}{x\sqrt{x^2+2x-1}}$

Let $z = x + \sqrt{x^2+2x-1}$

$dz = \left(1+\frac{2x+2}{2\sqrt{x^2+2x-1}}\right)dx$

$= \left(1+\frac{x+1}{\sqrt{x^2+2x-1}}\right)dx$

$dz = \left(\frac{\sqrt{x^2+2x-1}+x+1}{\sqrt{x^2+2x-1}}\right)dx$

$$I = \int \frac{1}{\frac{z^2+1}{2(z+1)}} \frac{dz}{(z+1)}$$

$$= 2\int \frac{dz}{1+z^2}$$

$= 2 \tan^{-1} z + c$

$= 2 \tan^{-1} (x + \sqrt{x^2+2x-1}) + c$

Sol.5 $I = \int \left[\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right] dx \ (\ell n\, x)$

put $\left(\frac{x}{e}\right)^x = t \Rightarrow x\, \ell n \left(\frac{x}{e}\right) = \ell n\, t$

$$\left(\ell n\left(\frac{x}{e}\right) + \frac{x}{\left(\frac{x}{e}\right)}.\frac{1}{e}\right) dx = \frac{dt}{t}$$

$$\left(1 + \ell n\left(\frac{x}{e}\right)\right) dx = \frac{dt}{t} \Rightarrow \ell n\, x\, dx = \frac{dt}{t}$$

$$I = \int \left(t + \frac{1}{t}\right)\frac{1}{t} dt$$

$$= \int\left(1+\frac{1}{t^2}\right) dt = t - \frac{1}{t} + c$$

$$= \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + c$$

Sol.6 $I = \int \frac{a^2 \sin^2 x + b^2 \cos^2 x}{a^4 \sin^2 x + b^4 \cos^2 x} dx$

$$= \frac{1}{(a^2+b^2)} \int \frac{(a^2+b^2)(a^2 \sin^2 x + b^2 \cos^2 x)}{(a^4 \sin^2 x + b^4 \cos^2 x)} dx$$

$$= \frac{1}{(a^2+b^2)} \int \frac{a^4 \sin^2 x + b^4 \cos^2 x + a^2b^2}{(a^4 \sin^2 x + b^4 \cos^2 x)} dx$$

$$= \frac{1}{(a^2+b^2)} \left[\int dx + a^2b^2 \int \frac{dx}{a^4 \sin^2 x + b^2 \cos^2 x}\right]$$

$$= \frac{1}{(a^2+b^2)} \left[x + a^2b^2 \int \frac{\sec^2 x\, dx}{b^4 + a^4 \tan^2 x}\right]$$

$$= \frac{1}{(a^2+b^2)} \left[x + \frac{a^2b^2}{a^4} \int \frac{\sec^2 x\, dx}{\left(\frac{b}{a}\right)^4 + \tan^2 x}\right]$$

$$= \frac{1}{(a^2+b^2)} \left[x + \frac{a^2b^2}{a^4\left(\frac{b}{a}\right)^2} \tan^{-1}\left(\frac{a^2 \tan x}{b^2}\right)\right] + c$$

$$= \frac{1}{(a^2+b^2)} \left[x + \tan^{-1}\left(\frac{a^2 \tan x}{b^2}\right)\right] + c$$

Sol.7 $I = \int \frac{dx}{\left(x + \sqrt{x^2-1}\right)^2}$

put $x + \sqrt{x^2-1} = t \Rightarrow \left(1 + \frac{2x}{2\sqrt{x^2-1}}\right) dx = dt$

$$\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right) dx = dt \Rightarrow \frac{t\, dx}{\sqrt{x^2-1}} = dt$$

$$I = \int \frac{\sqrt{x^2-1}}{t^3} dt = \int \left(\frac{t-x}{t^3}\right) dt$$

$$= \int \left(\frac{1}{t^2} - \frac{t^2+1}{2t^4}\right) dt = \int \left(\frac{1}{t^2} - \frac{1}{2t^2} - \frac{1}{2t^4}\right) dt$$

$$= \frac{1}{2}\int \frac{1}{t^2} dt - \frac{1}{2}\int \frac{dt}{t^4} = -\frac{1}{2t} + \frac{1}{6}\frac{1}{t^3} + c$$

$$= \frac{2x^3}{3} - x - \frac{2}{3}(x^2-1)^{3/2} + c$$

Sol.8 $I = \int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}} dx$

$$= \int \frac{\sin(x-a)}{\sqrt{\sin(x+a)\sin(x-a)}} dx$$

$$= \int \frac{\sin x \cos a - \cos x \sin a}{\sqrt{\sin^2 x - \sin^2 a}} dx$$

$$I = \cos a \underbrace{\int \frac{\sin x\, dx}{\sqrt{\sin^2 x - \sin^2 a}}}_{I_1} - \sin a \underbrace{\int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2 a}}}_{I_2}$$

$$I_1 = \int \frac{\sin x\, dx}{\sqrt{\sin^2 x - \sin^2 a}} = \int \frac{\sin x\, dx}{\sqrt{\cos^2 a - \cos^2 x}}$$

Let cos x = u $\Rightarrow$ sin x dx = – du

$$= -\int \frac{du}{\sqrt{\cos^2 a - u^2}} = \cos^{-1}\left(\frac{u}{\cos a}\right)$$

$$= \cos^{-1}(\cos x \sec a)$$

$$I_2 = \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2 a}}$$

Let sin x = v $\Rightarrow$ cos x dx = dv

$$= \int \frac{dv}{\sqrt{v^2 - \sin^2 a}} = \ell n\left(v + \sqrt{v^2 - \sin^2 a}\right) + c$$

$$= \ell n\left(\sin x + \sqrt{\sin^2 x - \sin^2 a}\right) + c$$

$$I = \cos a \cos^{-1}(\cos x \sec a) - \sin a$$

$$\ell n\left(\sin x + \sqrt{\sin^2 x - \sin^2 a}\right) + c$$

Sol.9 $I = \int \frac{\cot x\, dx}{(1-\sin x)(\sec x + 1)}$

$$= \int \frac{\cot x\, dx}{\frac{(1-\sin x)(1+\cos x)}{\cos x}}$$

$$= \int \frac{\cos^2 x}{\sin x(1-\sin x)(1+\cos x)}\, dx$$

$$= \int \frac{(1+\sin x)}{\sin x(1+\cos x)}\, dx$$

$$= \int \frac{dx}{\sin x(1+\cos x)} + \int \frac{dx}{(1+\cos x)}$$

$$= \int \frac{\left(1+\tan^2 \frac{x}{2}\right)^2}{2\tan\frac{x}{2}\left(1+\tan^2\frac{x}{2} + 1 - \tan^2\frac{x}{2}\right)}$$

$$+ \int \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\, dx$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)\sec^2\frac{x}{2}\, dx}{4\tan\frac{x}{2}} + \frac{1}{2}\int \sec^2\frac{x}{2}\, dx$$

$$\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\, dx = 2\, dt$$

$$= \frac{1}{2}\int\left(\frac{1+t^2}{t}\right) dt + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\ell n\, t + \frac{t^2}{4} + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\ell n \tan\frac{x}{2} + \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + c$$

Sol.10 $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

put x = a tan² θ $\Rightarrow$ dx = 2a tan θ sec² θ dθ

$$= \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a\sec^2\theta}}\quad (2a\tan\theta\sec^2\theta)d\theta$$

$$= 2a\int \underbrace{\theta}\ \underbrace{(\tan\theta\sec^2\theta)}\, d\theta$$

$$= 2a\,\theta\,\frac{\tan^2\theta}{2} - 2a\int\frac{\tan^2\theta}{2}\, d\theta$$

$$= a\theta\tan^2\theta - a\int(\sec^2\theta - 1)\, d\theta$$

$$= a\theta\tan^2\theta - a\tan\theta + a\theta + c$$

$$= (a+x)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c$$

Sol.11 $I = \int \frac{\sqrt{x^2+1}}{x^4}\, \ell n\left(\frac{x^2+1}{x^2}\right) dx$

$$= \int \sqrt{\frac{x^2+1}{x^2}}\ \ell n\left(\frac{x^2+1}{x^2}\right)\cdot\frac{1}{x^3}\, dx$$

$$= \int \sqrt{1+\frac{1}{x^2}}\ \ell n\left(1+\frac{1}{x^2}\right)\cdot\frac{1}{x^3}\, dx$$

$$1 + \frac{1}{x^2} = t \Rightarrow \frac{dx}{x^3} = -\frac{dt}{2}$$

$= -\frac{1}{2}\int \sqrt{t}\,\ell n\, t\, dt$

$= -\frac{1}{2}\left[\frac{2}{3}t\sqrt{t}\,\ell n\, t - \frac{2}{3}\int \sqrt{t}\, dt\right]$

$= -\frac{1}{2}\left[\frac{2}{3}t\sqrt{t}\,\ell n\, t - \frac{4}{9}t\sqrt{t}\right] + c$

$= -\frac{1}{2}\left[\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\ell n\left(1+\frac{1}{x^2}\right) - \frac{4}{9}\left(1+\frac{1}{x^2}\right)^{3/2}\right] + c$

$= \frac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left(2-3\,\ell n\left(1+\frac{1}{x^2}\right)\right) + c$

Sol.12 $I = \int \frac{x+1}{x(1+xe^x)^2}\,dx$

put $1 + xe^x = t \Rightarrow (xe^x + e^x)\,dx = dt$

$(x + 1)\,dx = \frac{dt}{e^x}$

$= \int \frac{dt}{xe^x\, t^2}$

$I = \int \frac{dt}{(t-1)\,t^2}$

$\frac{1}{(t-1)t^2} = \frac{A}{(t-1)} + \frac{B}{t} + \frac{C}{t^2}$

$I = At^2 + Bt\,(t-1) + C(t-1)$

$A = 1$; $B = -1$; $C = -1$

$I = \int \frac{dt}{t-1} - \int \frac{dt}{t} - \int \frac{dt}{t^2}$

$= \ell n\,\frac{t-1}{t} + \frac{1}{t} + c$

$= \ell n\,\frac{xe^x}{1+xe^x} + \frac{1}{1+xe^x} + c$

Sol.13 $\int \frac{f(x)}{x^2(x+1)^3}\,dx$

$= \int \left\{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}\right\}dx$

$\because \int \frac{f(x)}{x^2(x+1)^2}\,dx$ is a rational function

$\therefore$ A = 0 & C = 0

$\int \frac{f(x)}{x^2(x+1)^3}\,dx = \int \left\{\frac{B}{x^2} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}\right\}dx$

$\int \frac{f(x)}{x^2(x+1)^3}\,dx$

$= \int \frac{B(x+1)^3 + Dx^2(x+1) + E\cdot x^2}{x^2(x+1)^3}\,dx$

$f(x) = B(x + 1)^3 + Dx^2(x + 1) + Ex^2$

$f(x) = (B + D)x^3 + (3B + D + E)x^2 + 3Bx + B$

$\because$ f(x) is a quadratic function

$\therefore$ $f(x) = (3B + D + E)\,x^2 + 3Bx + B$

$\because$ $f(0) = 1 \Rightarrow B = 1$

Now $f'(x) = 2(3B + D + E)x + 3B$

$f'(0) = 3B$

$f'(0) = 3$

Sol.14 $f(x) = \tan^{-1} x + \ell n\,\sqrt{\frac{1+x}{1-x}}$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1-x^2}$

$f'(x) = \frac{2}{1-x^4} \Rightarrow \frac{1}{2}f'(x) = \frac{1}{1-x^4}$

$\int \frac{1}{1-x^4}\,d(x^4)$

$= -\ell n\,(1 - x^4) + c$

Sol.15 $I = \int \frac{(\sqrt{x}+1)}{\sqrt{x}\,(x^{1/3}+1)}\,dx$

put $x = t^6 \Rightarrow dx = 6t^5\,dt$

$= \int \frac{(t^3+1)}{t^3(t^2+1)}\,6t^5\,dt = 6\int \frac{t^2(t^3+1)}{(t^2+1)}\,dt$

$= 6\int \frac{(t^2+1-1)}{(t^2+1)}\,(t^3 + 1)dt$

$= 6\int (t^3+1)\,dt - 6\int \frac{t^3\,dt}{t^2+1} - 6\int \frac{dt}{t^2+1}$

$= 6\left(\frac{t^4}{4}+t\right) - 6\int \frac{t^2.t\,dt}{t^2+1} - 6\tan^{-1} t$

put $t^2 + 1 = z \Rightarrow tdt = \frac{dz}{2}$

$$= 6\left[\left(\frac{t^4}{4}+t\right)\right] - 3\int\left(\frac{z-1}{z}\right)dz \; - 6\tan^{-1}t + c$$

$$= 6\left[\frac{t^4}{4}+t\right] - 3z + 3\,\ell n\, z - 6\tan^{-1}t + c$$

$$= 6\left[\frac{t^4}{4}+t\right] - 3(t^2+1) + 3\,\ell n\,(1+t^2) - 6\tan^{-1}t + c$$

where $t = x^{1/6}$

Sol.16 $I = \int \frac{dx}{\sin\frac{x}{2}\sqrt{\cos^3\frac{x}{2}}}$

put $\frac{x}{2} = t \Rightarrow dx = 2dt$

$$= 2\int \frac{dt}{\sin t\sqrt{\cos^3 t}} = 2\int \frac{\sin t\, dt}{\sin^2 t\sqrt{\cos^3 t}}$$

$\cos t = z^2 \Rightarrow \sin t\, dt = -2z\, dz$

$$= -4\int \frac{z\, dz}{(1-z^4)\, z^3} = -4\int \frac{dz}{z^2(1-z^4)}$$

$$= 4\int \frac{dz}{z^2(z^2-1)(z^2+1)}$$

$$\frac{1}{z^2(z^2-1)(z^2+1)} = \frac{A}{z^2} + \frac{B}{z^2-1} + \frac{C}{z^2+1}$$

$I = A(z^2+1)(z^2-1) + B z^2(z^2-1) + Cz^2(z^2-1)$

$A = -1$; $C = 1/2$; $B = 1/2$

$$I = \left[-\int \frac{dz}{z^2} + \frac{1}{2}\int \frac{dz}{z^2+1} + \frac{1}{2}\int \frac{dz}{z^2-1}\right]$$

$$= \frac{4}{z} + 2\tan^{-1}z + \ell n\,\frac{z-1}{z+1} + c$$

$$= \frac{4}{\sqrt{\cos\frac{x}{2}}} + 2\tan^{-1}\sqrt{\cos\frac{x}{2}} - \ell n\,\frac{1+\sqrt{\cos\frac{x}{2}}}{1-\sqrt{\cos\frac{x}{2}}} + c$$

Sol.17 $I = \int \frac{x^2+x}{(e^x+x+1)^2}\,dx$

$$I = \int \frac{(x^2+x)\,dx}{e^{2x}\{1+(x+1)e^{-x}\}^2}$$

$$I = \int \frac{xe^{-x}.(1+x)e^{-x}}{\{1+(x+1)e^{-x}\}^2}\,dx$$

put $1 + (x+1)e^{-x} = t \Rightarrow xe^{-x}\,dx = -dt$

Now $I = -\int \frac{t-1}{t^2}\,dt \Rightarrow -\int \frac{dt}{t} + \int \frac{dt}{t^2}$

$$= -\ell n\, t - \frac{1}{t} + c$$

$$= -\ln\{1 + (x+1)e^{-x}\} - \frac{1}{1+(x+1)e^{-x}} + c$$

Sol.18 $I = \int \sqrt{\frac{\text{cosec } x - \cot x}{\text{cosec } x + \cot x}}\,.\frac{\sec x}{\sqrt{1+2\sec x}}.dx$

$$= \int \frac{1}{(\text{cosec } x + \cot x)}.\frac{\sec x}{\sqrt{1+2\sec x}}$$

$$= \int \frac{\tan x.\sec x}{(\sec x+1)\sqrt{1+2\sec x}}\,dx$$

Let $1 + 2\sec x = t^2$

$I = \int \frac{dt}{(t^2+1).t}$ (use partial fractions)

Sol.19 $I = \int \frac{\cos x - \sin x}{7 - 9\sin 2x}.dx$

$$= \int \frac{\cos x - \sin x}{16 - 9(\sin x + \cos x)^2}.dx$$

Let $\sin x + \cos x = t$

$(\cos x - \sin x)\,dx = dt$

Sol.20 $I = \int \frac{dx}{\sec x + \text{cosec } x}$

$$= \int \frac{\sin x\cos x\, dx}{(\sin x + \cos x)} = \frac{1}{2}\int \frac{2\sin x\cos x + 1 - 1}{(\sin x + \cos x)}\,dx$$

$$= \frac{1}{2}\int (\sin x + \cos x) - \frac{1}{2}\int \frac{dx}{(\sin x + \cos x)}$$

$$= \frac{1}{2}[\sin x - \cos x] - \frac{1}{2\sqrt{2}}\int \frac{dx}{\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)}$$

$$= \frac{1}{2}(\sin x - \cos x) - \frac{1}{2\sqrt{2}}\int \text{cosec}\left(x + \frac{\pi}{4}\right)dx$$

$$= \frac{1}{2}(\sin x - \cos x) - \frac{1}{2\sqrt{2}}\left[\ell n\,\frac{1-\cos(x+\pi/4)}{\sin(x+\pi/4)}\right] + c$$

Sol.21 $I = \int \frac{dx}{\sin x + \sec x}$

$= \int \frac{dx}{\sin x + \frac{1}{\cos x}} = \int \frac{2\cos x\, dx}{2 + \sin 2x}$

$= \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + \sin 2x}$

$= \int \frac{\cos x + \sin x}{3 - (1 - \sin 2x)}\, dx + \int \frac{\cos x - \sin x}{1 + (1 + \sin 2x)}\, dx$

$= \int \frac{\sin x + \cos x}{\underset{\substack{\downarrow \\ \sin x - \cos x = t}}{3 - (\sin x - \cos x)^2}} . dx + \int \frac{\sin x - \cos x}{\underset{\substack{\downarrow \\ \sin x - \cos x = t}}{1 + (\sin x + \cos x)^2}} . dx$

Sol.22 $I = \int \tan x . \tan 2x \tan 3x\, dx$

$\tan (x + 2x) = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$

$\tan 3x = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$

$\tan 3x - \tan x \tan 2x \tan 3x = \tan x + \tan 2x$
$\tan x \tan 2x \tan 3x = \tan 3x - \tan 2x - \tan x$

$I = \int (\tan 3x - \tan 2x - \tan x)\, dx$

$= \frac{1}{3}\, \ell n \sec 3x - \frac{1}{2}\, \ell n \sec 2x - \ell n \sec x + c$

Sol.23 $I = \int \left(\frac{3 + 4\sin x + 2\cos x}{3 + 2\sin x + \cos x}\right) dx$

Let $3 + 4 \sin x + 2 \cos x$
$= A(3 + 2 \sin x + \cos x) + B(2 \cos x - \sin x) + C$
$3 + 4 \sin x + 2 \cos x = 3A + C + \sin x(2A - B) + \cos x (A + 2B)$

$3A + C = 3$ ......(1)
$2A - B = 4$ ......(2)
$A + 2B = 2$ ......(3)

$A = 2$; $B = 0$; $C = -3$

$I = 2\int dx - 3\int \frac{dx}{3 + 2\sin x + \cos x}$

$= 2x - 3\int \frac{(1 + \tan^2 x/2)}{3(1 + \tan^2 \frac{x}{2}) + 4\tan\frac{x}{2} + 4}\, dx$

$= 2x - 3\int \frac{\sec^2 x/2}{2\tan^2 \frac{x}{2} + 4\tan\frac{x}{2} + 1 - \tan^2 \frac{x}{2}}\, dx$

$= 2x - \frac{3}{2}\int \frac{\sec^2 x/2}{\tan^2 \frac{x}{2} + 2\tan\frac{x}{2} + 1 + 1}\, dx$

$= 2x - \frac{3}{2}\int \frac{\sec^2 x/2}{\left(\tan\frac{x}{2} + 1\right)^2 + 1}\, dx$

$\frac{x}{2} = t \Rightarrow dx = 2dt$

$I = 2x - 3\int \frac{\sec^2 t}{(1 + \tan t)^2 + 1}\, dt$

$= 2x - 3 \tan^{-1} (1 + \tan t) + c$

$= 2x - 3 \tan^{-1} \left(1 + \tan \frac{x}{2}\right) + c$

Sol.24 convert into $e^x (f(x) + f'(x))$ form

$I = \int \frac{e^{\cos x}(x\sin^3 x + \cos x)}{\sin^2 x}\, dx$

$I = \int e^{\cos x}(x \sin x + \cot x . \mathrm{cosec}\, x)\, dx$

$I = \int \underset{\mathrm{I}}{x} . \underbrace{e^{\cos x} . \sin x}_{\mathrm{I}}\, dx + \int \underbrace{e^{\cos x}}_{\mathrm{I}} . \underbrace{\cot x\, \mathrm{cosec}\, x\, dx}_{\mathrm{I}}$

using by parts

$I = -\ xe^{\cos x} + \int 1 . e^{\cos x}\, dx + e^{\cos x}(-\mathrm{cosec}\, x) - \int e^{\cos x}$
$(-\sin x)\, (-\mathrm{cosec}\, x)\, dx$

$I = -\ xe^{\cos x} +$

$\int e^{\cos x}\, dx - e^{\cos x}\, \mathrm{cosec}\, x - \int e^{\cos x}\, dx$

$I = e^{\cos x} (-x - \mathrm{cosec}\, x) + c$

Sol.25 $I = \int \frac{(ax^2 - b)dx}{x\sqrt{c^2x^2 - (ax^2 + b)^2}}$

$= \int \frac{a - b/x^2}{\sqrt{c^2 - \left(ax + \frac{b}{x}\right)^2}}\, dx$

$ax + \frac{b}{x} = t \Rightarrow a - \frac{b}{x^2}\, dx = dt$

$= \int \frac{dt}{\sqrt{c^2 - t^2}} = \sin^{-1} \frac{t}{c} + c = \sin^{-1} \left(\frac{ax + b/x}{c}\right) + C$

Sol.26 $I = \int e^x \left(\frac{1-x^2}{(1-x)\sqrt{1-x^2}} + \frac{1}{(1-x)\sqrt{1-x^2}} \right).dx$

$$= \int e^x \left(\underbrace{\sqrt{\frac{1+x}{1-x}}}_{\downarrow\, f(x)} + \underbrace{\frac{1}{(1-x)\sqrt{1-x^2}}}_{\downarrow\, f'(x)} \right).dx$$

Sol.27 $I = \int \frac{x}{(x^2-1)^{3/2}}.\ell n\, x\, dx$

use by parts

$$=\ln x \int \frac{x}{(x^2-1)^{3/2}}dx - \int \left(\frac{1}{x} \int \frac{xdx}{(x^2-1)^{3/2}} \right)dx$$

$x^2 - 1 = t^2$

$xdx = tdt$

$$\int \left(\frac{x}{(x^2-1)^{3/2}} \right)dx = \int \frac{tdt}{t^3} = -\frac{1}{t} = \frac{-1}{\sqrt{x^2-1}}$$

$$I = \frac{-\ln x}{\sqrt{x^2-1}} + \int \left(\frac{1}{x\sqrt{x^2-1}} \right)dx$$

$$I = \frac{-\ln x}{\sqrt{x^2-1}} + \sec^{-1}x + c$$

Sol.28 $I = \int \sqrt{\frac{(1-\sin x)(2-\sin x)}{(1+\sin x)(2+\sin x)}}\, dx$

$$I = \int \frac{\cos x}{1+\sin x} \sqrt{\frac{2-\sin x}{2+\sin x}}\, dx$$

put $\frac{2-\sin x}{2+\sin x} = t^2 \Rightarrow \sin x = \frac{2(1-t^2)}{1+t^2}$

$$\Rightarrow \cos x\, dx = \frac{-8t\, dt}{(1+t^2)^2}$$

Now, $I = -\int \frac{8t^2}{(1+t^2)^2 \left\{ 1 + \frac{2-2t^2}{1+t^2} \right\}} dt$

$$= 8\int \frac{t^2\, dt}{(t^2-3)(t^2+1)}$$

$$= 8\int \left[\frac{3}{4} \frac{1}{(t^2-3)} + \frac{1}{4(t^2+1)} \right] dt$$

$$= 6\int \frac{dt}{t^2-3} + 2\int \frac{dt}{t^2+1}$$

$$= \frac{6}{2\sqrt{3}}\, \ell n \left| \frac{t-\sqrt{3}}{t+\sqrt{3}} \right| + 2\tan^{-1} t + c$$

$$= \sqrt{3}\, \ell n \left| \frac{\sqrt{\frac{2-\sin x}{2+\sin x}} - \sqrt{3}}{\sqrt{\frac{2-\sin x}{2+\sin x}} + \sqrt{3}} \right| + 2\tan^{-1}\left(\sqrt{\frac{2-\sin x}{2+\sin x}} \right) + c$$

$$= \sqrt{3}\, \ell n \left| \frac{\sqrt{2-\sin x} - \sqrt{6+3\sin x}}{\sqrt{2-\sin x} + \sqrt{6+3\sin x}} \right|$$

$$+ 2\tan^{-1}\left(\sqrt{\frac{2-\sin x}{2+\sin x}} \right) + c$$

Sol.29 $I = \int \frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2(x^2+1)^2}\, dx$

$$I = \int \frac{4x^3(x^2+1) + 4x(x^2+1) - 7(x^2+1)^2 + 12x^2}{x^2(x^2+1)^2}\, dx$$

$$I = 4\int \frac{x\,dx}{x^2+1} + 4\int \frac{dx}{x(x^2+1)} - 7\int \frac{dx}{x^2} + 12\int \frac{dx}{(x^2+1)^2}$$

$$I = 2\, \ell n\, |x^2+1| + 2\ell n \left| \frac{x^2}{x^2+1} \right| + \frac{7}{x} + 12I_1 \quad ....(1)$$

$I_1 = \int \frac{dx}{(x^2+1)^2}$ put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$I_1 = \int \cos^2\theta\, d\theta$$

$$= \int \left(\frac{1+\cos 2\theta}{2} \right) d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + c_1$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2} \frac{x}{(x^2+1)} + c_1$$

put I_1 in equation (1)

$$I = 4\,\ell n\, x + \frac{7}{x} + 6\tan^{-1} x + \frac{6x}{x^2+1} + c$$

Sol.30 $I = \int \frac{\sqrt{2-x-x^2}}{x^2}\,dx$

$$= \int \frac{1}{x^2}\sqrt{2-x-x^2}\,dx$$

use by parts.

$$= \sqrt{2-x-x^2}\int\frac{dx}{x^2} - \int \frac{-1-2x}{2\sqrt{2-x-x^2}}\left(\frac{-1}{x}\right)dx$$

$$= \frac{-1}{x}\sqrt{2-x-x^2} - \frac{1}{2}\underbrace{\int\frac{dx}{x\sqrt{2-x-x^2}}}_{I_1} - \underbrace{\int\frac{dx}{\sqrt{2-x-x^2}}}_{I_2}$$

$$I = \frac{-1}{x}\sqrt{2-x-x^2} - \frac{1}{2}I_1 - I_2$$

$I_1 = \int\frac{dx}{x\sqrt{2-x-x^2}}$ put $x = \frac{1}{t} \Rightarrow dx = \frac{-dt}{t^2}$

$$= -\int\frac{dt}{\sqrt{2t^2-t-1}} - \frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{\left(t-\frac{1}{4}\right)^2 - \frac{9}{16}}}$$

$$= -\frac{1}{\sqrt{2}}\,\ell n\left[t - \frac{1}{4} + \sqrt{2t^2-t-1}\right]$$

$$I_2 = \int\frac{dx}{\sqrt{2-x-x^2}} = \int\frac{dx}{\sqrt{\frac{9}{4}-\left(x+\frac{1}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{2x+1}{3}\right)$$

Sol.31 $\sqrt{\frac{x-\beta}{x-\alpha}} = t$

$$\Rightarrow \frac{1}{2}\frac{(\beta-\alpha)}{\sqrt{x-\beta}\,(x-\alpha)^{3/2}}.dx = dt$$

so $I = -\int\frac{2dt}{\alpha-\beta} = -\frac{2}{\alpha-\beta}\sqrt{\frac{x-\beta}{x-\alpha}} + C$

Sol.32 $I = \int\frac{dx}{\cos^3 x - \sin^3 x}$

$$I = \int\frac{dx}{(\cos x - \sin x)(1+\sin x\cos x)}$$

$$= \int\frac{dx}{(\cos x - \sin x)\left(1+\frac{\sin 2x}{2}\right)}$$

$\because$ $\sin 2x = (\sin x + \cos x)^2 - 1$

$$I = \int\frac{(\cos x - \sin x)\,dx}{(\cos x - \sin x)^2\left\{1+\frac{1}{2}(\sin x+\cos x)^2 - \frac{1}{2}\right\}}$$

$$I = 2\int\frac{(\cos x - \sin x)\,dx}{2-(\sin x+\cos x)^2\ \{1+(\sin x+\cos x)^2\}}$$

put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\,dx = dt$

$$I = 2\int\frac{dt}{(2-t^2)(t^2+1)} = \frac{2}{3}\int\frac{(2-t^2)+(1+t^2)}{(2-t^2)(1+t^2)}\,dt$$

$$I = \frac{2}{3}\int\frac{dt}{t^2+1} + \frac{2}{3}\int\frac{dt}{2-t^2}$$

$$= \frac{2}{3}\tan^{-1} t + \frac{2}{3}\times\frac{1}{2\sqrt{2}}\,\ell n\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right| + c$$

$$= \frac{2}{3}\tan^{-1}(\sin x + \cos x)$$

$$+ \frac{1}{3\sqrt{2}}\,\ell n\left|\frac{\sqrt{2}+(\sin x+\cos x)}{\sqrt{2}-(\sin x+\cos x)}\right| + c$$